International Journal of Theoretical Physics, Vol. 14, No. 6 (1975), pp. 373–378

Division Algebras and the Factorization of Einstein's Field Equations

J. CHARLES NICKERSON

Mississippi State University, Mississippi State, Mississippi 39762

Received: 15 *November* 1974

Abstract

A division algebra of highest possible dimension is the eight-dimensional Cayley algebra. This remarkable property of mathematics suggests an intimate fundamental connection between Cayley algebras and descriptions of the physical universe. On this basis, it is suggested that Einstein's field equations with the huge Λ proposed by this author elsewhere be factored by use of such an algebra. This factorization promises to yield Dirac-like spinor quantum wave equations.

1. *Introduction*

There is an intimate connection among quadratic forms, symmetric matrices, and division algebras (Paige & Swift, 1961). Mendel Sachs has utilized this connection in conjunction with the symmetry of the metric and stress-energy tensors of general relativity (Sachs, 1967–72). Sachs has achieved a quaternion factorization of Einstein's general relativistic field equations which seems to yield the electromagnetic field and to give a derived inertial mass for local bodies, in agreement with Mach's principle. A similar quaternion factorization process is also essentially what Dirac did in setting up his spinor wave equations for the electron. Dirac, in fact, found a quaternion factorization of the quadratic form $E^2 - p_x{}^2 - p_y{}^2 - p_z{}^2$ (Dirac, 1928, 1947; Schiff, 1968; Edmonds, 1974).

The suggestion here is to apply a similar factorization procedure to the large Λ field equations proposed by this author (Nickerson, 1975b);

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = -(8\pi\kappa/c^2)T_{\mu\nu} \tag{1}$$

Here κ is Newton's universal gravitational constant, $\sim 2/3 \times 10^{-10}$ N m^2/kg^2; c is 3×10^8 m/sec; and Λ is proposed to be huge, on the order of $c^3/\kappa\hbar$, or about 10^{69} m^{-2}, $\hbar$ being Planck's constant. Since, as discussed elsewhere (Nickerson, 1975c), the large Λ theory leads to quantum wave equations, one expects that a quaternion factorization of the large Λ equations will yield quantum wave equations of the Dirac form for various elementary "particles".

The preceding discussion deals with real quaternion factorization. However, a remarkable property of division algebras in general, of which complex numbers and quaternions are examples, is that one of eight dimensions has the largest dimensionality possible (Paige & Swift, 1961, p. 253). "Eight dimensions" means all elements, i.e., "numbers," of the algebra can be represented as linear combinations of eight basis "elements." The Cayley algebras are such eight-dimensional algebras (Paige & Swift, 1961, p. 253). One is tempted, then, to let the large Λ field equations be complex, thereby hopefully requiring the highest possible dimensionality in the division algebra used to factor them. That is, it is suggested here that a Cayley algebra be used to factor the large Λ field equations (1), where the equations (1) are now allowed to be complex. One simple way to make them complex is to postulate that Λ be complex.

In Sec. 2 we shall examine the intimate connection between quadratic forms and division algebras. The paper concludes with a discussion of some points of interest.

2. *Division Algebras and the Factorization of Quadratic Forms*

Quadratic forms are intimately related to geometry. One of the simplest such forms is the Pythagorean form which gives the square of the length in Euclidean geometry: $l^2 = x^2 + y^2 + z^2$. In the Riemannian geometries used in relativity theory, the square of length is assumed to have the quadratic form $\Sigma g_{\mu\nu} x^\mu x^\nu$, where $g_{\mu\nu}$ is the symmetric metric "tensor," the sum (Σ) is over the number of dimensions of the geometry, and μ and ν in x^μ or x^ν index the dimension: x^1 is x, x^2 is y, etc. (see any text on general relativity, e.g. Adler et al., 1975).

We are concerned here with the structure of Riemannian geometries, in particular with the metric $g_{\mu\nu}$. Thus, we are concerned with the structure of quadratic forms. In this section we shall examine a close connection between division algebras and the factorization of such forms. We shall see that the simple Pythagorean form itself exemplifies and requires the development of higher algebras for its factorization.

We begin with a one-dimensional quadratic form ax^2, where a is a real number. We ask whether this can be factored with real coefficients only. The answer is yes, as we have in particular

$$ax^2 = (ax)(x) \tag{2a}$$

and we have in general

$$ax^2 = (\beta x)(\gamma x) \tag{2b}$$

with β, γ any real numbers such that $\beta\gamma = a$. Here β and γ are the factorization coefficients in Eq. (2b), and a and 1 are the factorization coefficients in Eq.

(2a). So, on to two dimensions: We ask whether $ax^2 + bxy + cy^2$ can be factored with real factorization coefficients if a, b, and c are any three real numbers. The answer is no. Consider the special case $x^2 + y^2$. The factorization of $x^2 + y^2$ requires complex algebra, for reasons closely connected to the necessity for a complex algebra in finding roots of the equation $x^2 + 1 = 0$. With complex numbers, we factor $x^2 + y^2$ as $(x + iy)(x - iy)$. More generally, using a complex algebra, we can find complex factorization coefficients for the general two-dimensional quadratic form $ax^2 + bxy + cy^2$ where a, b, and c are allowed to be not only real, but more generally complex. This is all closely related to the theorem on the existence of zeros for an arbitrary polynomial in complex algebra.*

Now we consider three and four dimensions. Note first that the simple two-dimensional form $x^2 + y^2$ which requires complex algebra for factorization is just the square of distance in two-dimensional Euclidean geometry. So our basic question on factorization of quadratic forms is a basic question of geometry. In three dimensions the general quadratic form is

$$ax^2 + by^2 + cz^2 + dxy + eyz + fxz \tag{3a}$$

where a, b, c, d, e, f we take for now to be real numbers. With $d = e = f = 0$ and $a = b = c = 1$, we have the quadratic form which gives the square of distance in three dimensions: $x^2 + y^2 + z^2$. This form cannot be factored with complex algebra alone. The Irish physicist and mathematician W. Hamilton, in looking for a mathematical representation for geometric rotations in three dimensions, found in the early 1840's an algebra which can indeed factor $x^2 + y^2 + z^2$ in particular and Eq. (3a) in general (Tait, 1875; Bork, 1966; Edmonds, 1974; see also Schiff, 1968). Hamilton's algebra has four basis elements, $1, i, j$, and k, as opposed to the two, 1 and i, of complex algebra. Thus, Eq. (3a) can be factored as

$$ax^2 + by^2 + cz^2 + dxy + eyz + fxz = (\alpha x + \beta y + \gamma z)(\theta x + \kappa y + \lambda z) \tag{3b}$$

where α, β, γ, θ, κ, and λ are "real" linear combinations of Hamilton's four basis elements $1, i, j$, and k. "Real" here means the coefficients in the linear combinations are all real numbers. For example

$$\alpha = a_1 1 + a_2 i + a_3 j + a_4 k \tag{3c}$$

where a_1, a_2, a_3, a_4 are all real numbers. "Things" such as α, β, γ, etc. are called "quaternions," and the algebra we deal with here is called "quaternion algebra" (Edmonds, 1974). The real, complex, and quaternion algebras discussed here are all called "division algebras" (Paige & Swift, 1961, p. 253).

Now it turns out that the same quaternion algebra of Hamilton also factors the general four-dimensional quadratic form with real coefficients (Sachs, 1967-72; Paige & Swift, 1961):

$$\sum_{\mu,\nu=1}^{4} g_{\mu\nu} x^\mu x^\nu \tag{4a}$$

* This theorem is known as "The Fundamental Theorem of Algebra." See, for example, Paige & Swift (1961), p. 267.

where the $g_{\mu\nu}$'s are real. It is just such a factorization of the basic Lorentzian special relativistic form

$$t^2 - x^2 - y^2 - z^2 \tag{4b}$$

that led Dirac to his spin-$\frac{1}{2}$ spinor equations for the electron and positron. Dirac actually factored the quadratic form

$$E^2 - c^2 p_x{}^2 - c^2 p_y{}^2 - c^2 p_z{}^2 \tag{4c}$$

in his work (Dirac, 1928, 1947; Schiff, 1968; Edmonds, 1974). Those familiar with special relativity will recognize Eq. (4b) as the invariant interval, analogous to the square of Euclidean distance in three dimensions, and Eq. (4c) as the magnitude of the four-vector momentum. (See any special relativity text, e.g., Taylor & Wheeler, 1966). Mendel Sachs, in important recent work (Sachs, 1967–72), has carried out an analogous factorization program on the general quadratic (4a), more specifically on the Einstein field equation (1) in "preferred" form (Nickerson, 1975b), i.e., with $\Lambda = 0$. Note that, because of the intimate relationship between quadratic forms and symmetric matrices (Paige & Swift, 1961), factorization of the quadratic form leads to a factorization of the symmetric matrix. Since Einstein's field equations involve only symmetric tensors, such a factorization of them would seem to be suggested. It is just such a program, on the $\Lambda = 0$ equations, that Sachs has undertaken. He has found some intriguing and promising results: a possibly unified electromagnetic-gravitational field theory, a Machian derivation of mass from the nonlinear field equations, and other results including profound implications for elementary "particle" theory (Sachs, 1967–72).

One has seen here, I hope, an intimate connection between the basic quadratic forms of geometry and the algebras of real, complex, and quaternion "numbers." The algebras serve to factor the quadratic forms of one, two, and three or four dimensions, respectively. The well-known connection between quadratic forms and symmetric matrices then leads to analogous factorization of the matrices by these algebras, and in particular to Dirac's spin-$\frac{1}{2}$ wave equation and to Sachs' more general work on factoring Einstein's "preferred" equations.

To apply the factorization to the "big Λ" field equations proposed by this author elsewhere (Nickerson, 1975b, c) would straight away seem a very interesting thing to do. One would guess, based on how quantum wave equations seem to come from the "big Λ" theory (Nickerson, 1975c), that such an application might yield the Dirac-type quantum equations for elementary "particles" as well as Sachs' results, but now quantized, with no assumptions other than the three classical principles of the "big Λ" theory. This is the program proposed here, with a slight, but significant, modification to be discussed in Sec. 3.

3. *Discussion*

The modification of the factorization program proposed here is to use an eight-dimensional Cayley algebra, with a possibly complex Λ, as mentioned in the Introduction. This is suggested by the remarkable property that the highest dimension possible for a division algebra is eight (Paige & Swift, 1961, p. 253). The suggested program, then, is to try to factor the complex "big Λ" field equations with a Cayley algebra, in a manner similar to (Sachs', 1967-72) factorization of Einstein's "preferred" equations with a real quaternion algebra. This program is expected to lead to Dirac-like quantum wave equations for elementary "particles," as indicated in (Nickerson, 1975c). In these wave equations, Λ^{-1} plays the role of Planck's constant $\hbar$.

The eightfold maximum on the dimensionality of division algebras suggests some other fundamental connections. It might be intimately connected with the four-dimensionality of the universe and with the four quantum numbers necessary for electrons to meet the requirements of Pauli's exclusion principle (Nickerson, 1975a). It may be that the eight elements of a Cayley algebra which factors Eq. (1) are intimately related to the eight generators of the group "$SU(3)$" and the "eightfold way" of M. Gell-Mann and Y. Neeman (Chew et al., 1964). Cayley algebras have distinctly different right and left inverses, a property suggestive of parity breaking and handedness (Atchison, 1974). These suggested connections need to be checked out in future investigations. They are very speculative at this point, but they are intriguing possibilities, and their detailed investigation should teach us a considerable amount about fundamental processes.

Acknowledgment

For any valid connections or effective presentations made here, credit is due to the grace of God.

References

Adler, R., Bazin, M., and Schiffer, M. (1975). *Introduction to General Relativity*, 2nd ed. McGraw-Hill, New York.
Atchison, T. A. (1974). Private communication.
Bork, A. (1966). *American Journal of Physics*, **34**, 202; and references therein.
Chew, G. F., Gell-Mann, M., Rosenfeld, A. H. (1964). *Scientific American*. **210**, No. 2, 74.
Dirac, P. A. M. (1928). *Proceedings of the Royal Society (London)*, **A117**, 610.
Dirac, P. A. M. (1947). *The Principles of Quantum Mechanics*, 3rd ed, Ch. XI. Oxford University Press, New York.
Edmonds, J. D., Jr. (1974). *American Journal of Physics*, **42**, 220; and references to works on quaternions and quaternion factorization therein.
Nickerson, J. C. (1975a). *International Journal of Theoretical Physics*, **13**, 337.
Nickerson, J. C. (1975b). *International Journal of Theoretical Physics*, **14**, 367.
Nickerson, J. C. (1975c). *International Journal of Theoretical Physics*, **14**, 379.
Paige, L. J. and Swift, J. D. (1961). *Elements of Linear Algebra*, Chaps. 8 and 9, especially p. 253, Ginn and Company, Boston.
Sachs, M. (1967). *Nuovo Cimento*, **47A**, 759.
Sachs, M. (1968). *Nuovo Cimento*, **55B**, 199.
Sachs, M. (1969). *Physics Today*, **22**, 51.
Sachs, M. (1970). *Nuovo Cimento*, **66B**, 137.

Sachs, M. (1971). *International Journal of Theoretical Physics,* 4, 433, which includes a comprehensive list of Sachs' works.
Sachs, M. (1972). *International Journal of Theoretical Physics,* 5, 161.
Schiff, L. I. (1968). *Quantum Mechanics,* 3rd ed., Section 52. McGraw-Hill, New York.
Tait, P. G. (1875). *An Elementary Treatise on Quaternions.* Oxford University Press, Oxford.
Taylor, E. F. and Wheeler, J. A. (1966). *Spacetime Physics,* p. 111 ff. W. H. Freeman, San Francisco.